Journal of Statistical Physics, Vol. 71, Nos. 3/4, 1993

Maximal Lyapunov Exponent and Rotation Numbers for Two Coupled Oscillators Driven by Real Noise*

N. Sri Namachchivaya[1] **and H. J. Van Roessel**[2]

Received July 28, 1992

Asymptotic expansions for the exponential growth rate, known as the Lyapunov exponent, and rotation numbers for two coupled oscillators driven by real noise are constructed. Such systems arise naturally in the investigation of the stability of steady-state motions of nonlinear dynamical systems and in parametrically excited linear mechanical systems. Almost-sure stability or instability of dynamical systems depends on the sign of the maximal Lyapunov exponent. Stability conditions are obtained under various assumptions on the infinitesimal generator associated with real noise provided that the natural frequencies are noncommensurable. The results presented here for the case of the infinitesimal generator having a simple zero eigenvalue agree with recent results obtained by stochastic averaging, where approximate Itô equations in amplitudes and phases are obtained in the sense of weak convergence.

KEY WORDS: Lyapunov exponents; rotation numbers; almost-sure stability; Itô equations; stochastic averaging.

1. INTRODUCTION

The almost-sure asymptotic stability of a linear stochastic differential equation depends on the sign of the maximal Lyapunov exponent. In the case of white noise, a general method for obtaining necessary and sufficient conditions for stability was presented by Khasminskii,[4] with explicit results for second-order systems obtained by Kozin and Prodromou[6] and Mitchell and Kozin.[7] A complete study of second-order systems, taking into consideration all possible singularities that can exist in one-dimensional diffusion, was presented by Nishioka.[8] However, there are very

* Dedicated to Thomas K. Caughey on the occasion of his 65th birthday.

[1] Department of Aeronautical and Astronautical Engineering, University of Illinois, Urbana, Illinois.

[2] Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada.

0022-4715/93/0500-0549$07.00/0

few results for the case in which the noise is ergodic and nonwhite. These results are due to Arnold *et al.*[3] and Pardoux and Wihstutz,[10] where almost-sure asymptotic stability of a second-order system with a colored noise process was considered. A survey of known results on the asymptotics of Lyapunov exponents and rotation number for general two-dimensional systems driven by white or real noise is given by Pinsky and Wihstutz.[11] More recently, Sri Namachchivaya[12] obtained analytic results for almost-sure asymptotic stability of multi-degree-of-freedom linear systems excited by combined stochastic and harmonic excitation. These results were derived under the assumption that one of the degrees of freedom is critically damped, while the remaining degrees of freedom are strongly stable.

In most studies reported to date, the necessary and sufficient conditions were obtained only for a second-order system or a multi-degree-of-freedom system with a single critical mode. However, to understand most physical phenomena, it is necessary to obtain results for multi-degree-of-freedom systems. The level of mathematical difficulty encountered in this type of analysis, due to the associated multidimensional diffusion problem, has restricted major developments in this area. Recently, the authors[14] have presented a perturbative theoretical approach to calculating the maximal Lyapunov exponent for stochastically coupled two-degree-of-freedom systems in which the noise was assumed to be white and of small intensity.

Often the external fluctuations are modeled by a zero-mean Gaussian white noise. Unfortunately, however, this does not always provide a good description of the fluctuations that occur in nature. In particular, the δ-correlation of the white noise is an idealization of the correlation of real processes, which often have finite correlation times. There are many stochastic processes which can describe such excitations. A simple process exhibiting such a finite correlation is an Ornstein–Uhlenbeck process, which is a solution of the stochastic differential equation

$$du = -\alpha u\,dt + \sqrt{2}\,\sigma\,dW$$

where the correlation function and the spectral density are given by

$$R(\tau) = \frac{\sigma^2}{\alpha} e^{-\alpha|\tau|}, \qquad S(\omega) = \frac{\sigma^2}{\alpha^2 + \omega^2}$$

It is worth noting that by scaling $\sigma = \sqrt{\varepsilon}\,\alpha$ and letting α tend to ∞, the noise becomes standard white noise and the parameter α describes the bandwith of the noise, while ε represents its spectral density.

In this paper, the results of ref. 14 for white noise are extended to incorporate real noise excitation with specific infinitesimal generators G.

The systems under consideration consist of multiplicative two-degree-of-freedom systems. Such systems are encountered in the study of mechanical systems subjected to fluctuating loading or imposed displacements as well as in the investigation of the stability of steady-state motions of nonlinear dynamical systems. Consider, for example,

$$\ddot{y}_1 + 2\zeta\omega_1 \dot{y}_1 + \omega_1^2 y_1 + f(y_1, y_2) = w(t)$$
$$\ddot{y}_2 + 2\zeta\omega_2 \dot{y}_2 + \omega_2^2 y_2 + y_2 g(y_1, y_2) = 0$$

where $f(0, 0) = 0$, $\partial f/\partial y_2(y_1, 0) \neq 0$, and $w(t)$ is a white noise process. Then the stability of the solution ($y_1 = \xi(t)$, $y_2 = 0$) is governed by a set of variational equations (1) with $\xi(t)$ defined by

$$\ddot{\xi} + 2\zeta\omega_1 \dot{\xi} + \omega_1^2 \xi + f(\xi, 0) = w(t)$$

The purpose of this work is to develop approximations for the top Lyapunov exponents under various hypotheses about the real noise process. When G has an isolated simple zero eigenvalue, the results for the maximal Lyapunov exponent obtained here agree with those of Ariaratnam and Xie[1] and Sri Namachchivaya and Talwar.[13] In addition, an approximation of the rotation number for each degree of freedom is also developed. The problem is formulated in Section 2. An asymptotic expansion leads to the general results appearing in Section 3. Section 4 contains an explicit evaluation of the Lyapunov exponent and rotation numbers. Section 5 contains some concluding remarks.

2. STATEMENT OF THE PROBLEM AND THE FORMULATION

Consider linear oscillatory systems described by equations of motion of the form

$$\ddot{q}_i + \omega_i^2 q_i + \varepsilon^2 2\zeta\omega_i \dot{q}_i + \varepsilon \sum_{j=1}^{2} k_{ij} q_j f(\xi(t)) = 0, \qquad i, j = 1, 2 \tag{1}$$

where the q_i's are generalized coordinates, ω_i is the ith natural frequency, and $\varepsilon\zeta$ represents a small viscous damping coefficient. It is assumed that the natural frequencies are *noncommensurable.* The stochastic term $\varepsilon\xi(t)$ is a small-intensity, real-noise process on a smooth connected Riemannian manifold M (with or without boundary). The smooth nonconstant function $f: M \to R$ is such that $f(\xi(t))$ has zero mean. The associated infinitesimal generator is assumed to have the form

$$G(\xi) = \sum_{i=1}^{n} \mu_i(\xi) \frac{\partial}{\partial \xi_i} + \frac{1}{2} \sum_{k=1}^{m} \left[\sum_{i=1}^{n} \sigma_i^k(\xi) \frac{\partial}{\partial \xi_i} \right] \left[\sum_{j=1}^{n} \sigma_j^k(\xi) \frac{\partial}{\partial \xi_j} \right] \tag{2}$$

For the case where M is one-dimensional, the solution of the associated adjoint problem $G^*v(\xi)=0$ can explicitly be written in terms of scale and speed measures as

$$v(\xi)=m(\xi)[c_1 S(\xi)+c_2] \tag{3a}$$

where

$$S(\xi)=\int^{\xi} s(\eta)\, d\eta, \qquad m(\xi)=[\sigma^2(\xi)\, s(\xi)]^{-1}$$

$$s(x)=\exp\left[-\int^{x}\frac{2\tilde{\mu}(\eta)}{\sigma^2(\eta)}\, d\eta\right], \qquad \tilde{\mu}(\xi)=\mu(\xi)+\frac{1}{2}\sigma(\xi)\frac{\partial\sigma(\xi)}{\partial\xi}$$

and the constants c_1 and c_2 are obtained by using the boundary and normality conditions. The solution of $Gu(\xi)=0$ is

$$u(\xi)=\alpha+\beta S(\xi) \tag{3b}$$

In order to make the problem tractable, *G will be assumed to have an isolated simple zero eigenvalue.* Hence, the only solution for $Gu=0$ is $u\equiv\text{const}$ (i.e., there are no other linearly independent solutions). From this, it follows that the associated adjoint operator G^* also has zero as a simple, isolated eigenvalue and the normalized invariant measure $v(\xi)\, d\xi$ satisfies $G^*v(\xi)=0$. For the case of one-dimensional diffusion, the above assumption leads to the natural boundary condition $\sigma=0$ on ∂M and the zero flux property

$$-\tilde{\mu}(\xi)\, v(\xi)+\frac{1}{2}\frac{\partial}{\partial\xi}(\sigma^2(\xi)\, v(\xi))=0$$

The almost-sure stability of the equilibrium state $q=\dot{q}=0$ of Eq. (1) is to be investigated. Using the transformations $q_i=x_{2i-1}$, $\dot{q}_i=\omega_i x_{2i}$, $i=1, 2$, we can represent Eq. (1) as the following system of Stratonovich differential equations:

$$\begin{aligned} \dot{x}&=Ax+f(\xi(t))\, Bx \\ d\xi&=\mu(\xi)\, dt+\sigma(\xi)\circ dW_t \end{aligned} \tag{4}$$

where A and B are 4×4 constant matrices given by

$$A=\begin{bmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & -\varepsilon^2 2\zeta\omega_1 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & -\varepsilon^2 2\zeta\omega_2 \end{bmatrix}$$

$$B = \varepsilon \begin{bmatrix} 0 & 0 & 0 & 0 \\ -k_{11}/\omega_1 & 0 & -k_{12}/\omega_1 & 0 \\ 0 & 0 & 0 & 0 \\ -k_{21}/\omega_2 & 0 & -k_{22}/\omega_2 & 0 \end{bmatrix}$$

and W_t is the standard Wiener process. According to Oseledec's[7] multiplicative ergodic theory (see Arnold and Wihstutz[2]), for a stationary stochastic process and initial random variable $x_0 \neq 0$, the Lyapunov exponent (i.e., the exponential growth rate) of the corresponding solution $x(t; x_0)$ of Eq. (4) is given by

$$\lambda(x_0) = \lim_{t \to \infty} \frac{1}{t} \log \|x(t; x_0)\| \qquad \text{w.p. } 1$$

Moreover, for ergodic random processes, the random variable $\lambda(x_0)$ takes on nonrandom values

$$\lambda^{\min} = \lambda_p < \lambda_{p-1} < \cdots < \lambda_1 = \lambda^{\max}$$

with nonrandom multiplicities which add up to 4 in our case. The top Lyapunov exponent $\lambda^{\max}$, which will be denoted as λ henceforth, defines the sample stability or instability of the trivial solution of Eq. (4). However, the explicit evaluation of λ in terms of system parameters, which, as we shall show in this section, is given by an integral over the unit sphere with respect to the invariant measure of an associated diffusion, is usually difficult. Due to this, we only present an asymptotic expansion of λ in terms of the intensity of the noise and damping, which are assumed small. To this end, the usual transformation is introduced to the original system (4), i.e.,

$$x_{2i-1} = r_i \cos \phi_i, \qquad x_{2i} = -r_i \sin \phi_i, \qquad \rho_i = \ln(r_i)$$

which yields

$$\dot{\rho}_i = \varepsilon^2[\tilde{p}_i(\phi)] + \varepsilon[p_i(\exp(\rho_j - \rho_i), \phi)]\, f(\xi(t))$$
$$\dot{\phi}_i = [\omega_i + \varepsilon^2 \tilde{h}_i(\phi)] + \varepsilon[h_i(\exp(\rho_j - \rho_i), \phi)]\, f(\xi(t))$$

where p_i and h_i contain terms of the form $\exp(\rho_j - \rho_i)$ for $i \neq j$. These terms are due to the stochastic coupling between the first and the second degrees of freedom. Since $\exp(\rho_j - \rho_i) = r_j/r_i$ is always positive, one can introduce a one-to-one mapping $\exp(\rho_2 - \rho_1) = \tan\theta$, where $0 \leqslant \theta \leqslant \pi/2$. This motivates the following transformation:

$$x_1 = r \cos \phi_1 \cos \theta, \qquad x_3 = r \cos \phi_2 \sin \theta$$

$$x_2 = -r \sin \phi_1 \cos \theta, \qquad x_4 = -r \sin \phi_2 \sin \theta$$

where

$$0 \leqslant \phi_i \leqslant 2\pi, \qquad 0 \leqslant \theta \leqslant \pi/2$$

In the above transformation, r represents the norm of the response, ϕ_1 and ϕ_2 are the angles of the first and second oscillators, respectively, and θ describes the coupling or exchange of energy between the first and second oscillators. Letting $\rho = \ln \|x\|$, we can write the Stratonovich stochastic differential equations for ρ, ϕ_1, ϕ_2, θ, and ξ as

$$\begin{aligned}
\dot{\rho} &= \varepsilon^2 \tilde{q}_1(\phi_1, \phi_2, \theta) + \varepsilon f(\xi(t))\, q_1(\phi_1, \phi_2, \theta) \\
\dot{\theta} &= \varepsilon^2 \tilde{q}_2(\phi_1, \phi_2, \theta) + \varepsilon f(\xi(t))\, q_2(\phi_1, \phi_2, \theta) \\
\dot{\phi}_i &= \omega_i + \varepsilon^2 \tilde{h}_i(\phi_1, \phi_2, \theta) + \varepsilon f(\xi(t))\, h_i(\phi_1, \phi_2, \theta), \qquad i = 1, 2 \\
d\xi &= \mu(\xi)\, dt + \sigma(\xi) \circ dW_t
\end{aligned} \tag{5}$$

where

$$\tilde{q}_1 = -\left[\eta_1(1 - \cos 2\phi_1) \cos^2 \theta + \eta_2(1 - \cos 2\phi_2) \sin^2 \theta\right]$$

$$\tilde{q}_2 = \frac{1}{2}\left[\eta_1(1 - \cos 2\phi_1) - \eta_2(1 - \cos 2\phi_2)\right] \sin 2\theta$$

$$q_1 = \frac{1}{2}\left[(p_{21} \cos \phi_1 \sin \phi_2 + p_{12} \sin \phi_1 \cos \phi_2) \sin 2\theta + p_{22} \sin 2\phi_2 \sin^2 \theta + p_{11} \sin 2\phi_1 \cos^2 \theta\right]$$

$$q_2 = \frac{1}{4}\left[(p_{22} \sin 2\phi_2 - p_{11} \sin 2\phi_1) \sin 2\theta + 4p_{21} \cos \phi_1 \sin \phi_2 \cos^2 \theta - 4p_{12} \sin \phi_1 \cos \phi_2 \sin^2 \theta\right]$$

$$h_1 = p_{11} \cos^2 \phi_1 + p_{12} \cos \phi_1 \cos \phi_2 \tan \theta$$

$$h_2 = p_{22} \cos^2 \phi_2 + p_{21} \cos \phi_1 \cos \phi_2 \cot \theta$$

$$\tilde{h}_i = -\eta_i \sin 2\phi_i, \qquad p_{ij} = \frac{k_{ij}}{\omega_i}, \qquad \eta_i = \zeta \omega_i$$

Since the processes $(\phi_1, \phi_2, \theta, \xi)$ do not depend on ρ, the processes $(\phi_1, \phi_2, \theta, \xi)$ alone form a diffusive Markov process and the associated generator is given by

$$L^\varepsilon = L^0 + \varepsilon L^1 + \varepsilon^2 L^2 \tag{6}$$

where

$$L^0 = \sum_{i=1}^{2} \omega_i \frac{\partial}{\partial \phi_i} + G(\xi)$$

$$L^1 = f(\xi)\left(q_2 \frac{\partial}{\partial \theta} + \sum_{i=1}^{2} h_i \frac{\partial}{\partial \phi_i}\right)$$

$$L^2 = \tilde{q}_2 \frac{\partial}{\partial \theta} + \sum_{i=1}^{2} \tilde{h}_i \frac{\partial}{\partial \phi_i}$$

Integrating Eq. (5a) for ρ yields

$$\|x(t; x_0)\| = \|x_0\| \exp\left[\int_0^t Q^\varepsilon(\varphi_1(\tau), \varphi_2(\tau), \theta(\tau), \xi(\tau))\, d\tau\right] \tag{7}$$

where

$$Q^\varepsilon(\varphi_1, \varphi_2, \theta, \xi) = \varepsilon f(\xi)\, q_1(\varphi_1, \varphi_2, \theta) + \varepsilon^2 \tilde{q}_1(\varphi_1, \varphi_2, \theta)$$

According to Oseledec's[7] multiplicative ergodic theory, under the assumption that L^ε is ergodic, the top Lyapunov exponent is given by

$$\lambda^\varepsilon = \langle Q^\varepsilon, p^\varepsilon \rangle \equiv \int_0^{\pi/2} \int_M \int_0^{2\pi} \int_0^{2\pi} Q^\varepsilon(\phi_1, \phi_2, \theta, \xi) \times p^\varepsilon(\phi_1, \phi_2, \theta, \xi)\, d\phi_1\, d\phi_2\, d\xi\, d\theta \tag{8}$$

where p^ε is the unique ergodic invariant probability measure given by

$$L^{\varepsilon *} p^\varepsilon = 0 \tag{9}$$

provided L^ε is hypoelliptic. In addition, the rotation number for each degree of freedom can be written as

$$\alpha_i^\varepsilon = \lim_{t \to \infty} \frac{1}{t} \tan^{-1}\left(-\frac{x_{2i}}{x_{2i-1}}\right) = \langle H_i^\varepsilon, p^\varepsilon \rangle \tag{10}$$

where

$$H_i^\varepsilon(\varphi_1, \varphi_2, \theta, \xi) = \omega_i + \varepsilon f(\xi)\, h_i(\varphi_1, \varphi_2, \theta) + \varepsilon^2 \tilde{h}_i(\varphi_1, \varphi_2, \theta)$$

3. ASYMPTOTIC ANALYSIS

In order to obtain the solution given by the integral (8), we expand

$$Q^\varepsilon = Q^0 + \varepsilon Q^1 + \varepsilon^2 Q^2$$

and construct a formal expansion of $p^\varepsilon = p^0 + \varepsilon p^1 + \cdots + \varepsilon^N p^N + \cdots$ so that the Fokker–Planck equation

$$L^{\varepsilon *} p^\varepsilon = (L^0 + \varepsilon L^1 + \varepsilon^2 L^2)(p^0 + \varepsilon p^1 + \cdots) = 0$$

yields a sequence of problems for p^0, p^1, p^2,...:

$$L^{0*} p^0 = 0, \qquad L^{0*} p^1 = -L^{1*} p^0, \qquad L^{0*} p^2 = -L^{1*} p^1 - L^{2*} p^0, \ldots \tag{11}$$

By constructing an expansion for the adjoint problem $L^\varepsilon F^\varepsilon = Q^\varepsilon$, it will be shown that the expansion for the maximal Lyapunov exponent

$$\begin{aligned} \lambda^\varepsilon = {} & \langle Q^0, p^0 \rangle + \varepsilon(\langle Q^0, p^1 \rangle + \langle Q^1, p^0 \rangle) \\ & + \varepsilon^2(\langle Q^0, p^2 \rangle + \langle Q^1, p^1 \rangle + \langle Q^2, p^0 \rangle) + \cdots \end{aligned} \tag{12a}$$

is in fact asymptotic. To this end, as in Arnold *et al.*,[3] consider an adjoint expression for $L^\varepsilon F^\varepsilon = Q^\varepsilon$ with $F^\varepsilon = F^0 + \varepsilon F^1 + \cdots + \varepsilon^N F^N$ such that

$$\begin{aligned} & (L^0 + \varepsilon L^1 + \varepsilon^2 L^2)(F^0 + \varepsilon F^1 + \cdots + \varepsilon^N F^N) \\ & \quad = Q^\varepsilon - (q^0 + \varepsilon q^1 + \cdots + \varepsilon^N q^N) + \varepsilon^{N+1}\{L^1 F^N + L^2 F^{N-1}\} \\ & \quad + \varepsilon^{N+2}\{L^2 F^N\} \end{aligned}$$

Here $q^0, q^1, \ldots, q^N$ are functions that do not depend on φ and θ, and are chosen so that the sequence of problems

$$L^0 F^0 = Q^0 - q^0, \qquad L^0 F^1 = Q^1 - q^1 - L^1 F^0$$
$$L^0 F^2 = Q^2 - q^2 - L^1 F^1 - L^2 F^0, \ldots, \qquad L^0 F^N = -q^N - L^1 F^{N-1} - L^2 F^{N-2}$$

is solvable. Then, defining $\tilde{p}^\varepsilon = p^0 + \varepsilon p^1 + \cdots + \varepsilon^N p^N$, and assuming that the marginal of both p^ε and $\tilde{p}^\varepsilon$ on M is $\upsilon(\xi)$, leads to the following error between the Lyapunov exponent defined in Eq. (8) and the expansion in Eq. (12a):

$$\begin{aligned} & \langle Q^\varepsilon, p^\varepsilon \rangle - \{\langle Q^0, p^0 \rangle + \varepsilon[\langle Q^0, p^1 \rangle + \langle Q^1, p^0 \rangle] \\ & \qquad + \varepsilon^2[\langle Q^0, p^2 \rangle + \langle Q^1, p^1 \rangle + \langle Q^2, p^0 \rangle] + \cdots \\ & \qquad + \varepsilon^N[\langle Q^0, p^N \rangle + \langle Q^1, p^{N-1} \rangle + \langle Q^2, p^{N-2} \rangle]\} \\ & \quad = -\varepsilon^{N+1}[\langle L^1 F^N + L^2 F^{N-1}, p^\varepsilon \rangle + \langle L^{1*} p^N + L^{2*} p^{N-1}, F^\varepsilon \rangle \\ & \qquad - \langle L^1 F^N + L^2 F^{N-1}, \tilde{p}^\varepsilon \rangle - \langle Q^1, p^N \rangle - \langle Q^2, p^{N-1} \rangle] \\ & \qquad - \varepsilon^{N+2}[\langle L^2 F^N, p^\varepsilon \rangle + \langle L^{2*} p^N, F^\varepsilon \rangle - \langle L^2 F^N, \tilde{p}^\varepsilon \rangle - \langle Q^2, p^N \rangle] \end{aligned}$$

Suppose that $p^0, p^1, \ldots, p^N$ and $F^0, F^1, \ldots, F^N$ are such that inner products on the rhs of the above equations are well defined and, due to the fact that p^ε is unknown, assume the existence of the following bounds:

$$\sup_{\varphi,\theta,\xi} |L^1F^N + L^2F^{N-1}| \leqslant K_1 < \infty, \qquad \sup_{\varphi,\theta,\xi} |L^2F^N| \leqslant K_2 < \infty$$

Then it is clear, using the above estimate, that the asymptotic expansion for a fixed $N \geqslant 0$ given by Eq. (12a) is valid. Since $Q^0 \equiv 0$, the approximation reduces to

$$\lambda^\varepsilon = \varepsilon\langle Q^1, p^0\rangle + \varepsilon^2(\langle Q^1, p^1\rangle + \langle Q^2, p^0\rangle) + O(\varepsilon^3) \tag{12b}$$

where p^0 and p^1 are governed by Eqs. (11) and satisfy the boundary conditions

$$\begin{aligned} p^0(\varphi_1 + 2\pi, \varphi_2, \theta, \xi) = p^0(\varphi_1, \varphi_2 + 2\pi, \theta, \xi) = p^0(\varphi_1, \varphi_2, \theta, \xi) \\ p^1(\varphi_1 + 2\pi, \varphi_2, \theta, \xi) = p^1(\varphi_1, \varphi_2 + 2\pi, \theta, \xi) = p^1(\varphi_1, \varphi_2, \theta, \xi) \end{aligned} \tag{13}$$

It is important to point out that in addition to smooth functions, p^0 and p^1 can be generalized functions due to certain singularities. In this work, all possible solutions of p^0 will be determined in order to calculate the leading-order approximation of the maximal Lyapunov exponent. It is clear from the first of Eqs. (11) that the general solution of p^0 is of the form

$$p^0(\varphi_1, \varphi_2, \theta, \xi) = \upsilon(\xi)\, \tilde{F}(\theta, \omega_1\varphi_2 - \omega_2\varphi_1) \exp\left[\frac{\Lambda(\theta)}{2\omega_1\omega_2}(\omega_1\varphi_2 + \omega_2\varphi_1)\right] \tag{14}$$

Since it is assumed that the frequencies ω_1 and ω_2 are noncommensurable, as shown in the Appendix, the periodic boundary conditions imply $\Lambda(\theta) = 0$ and $\tilde{F}(\theta, \omega_1\varphi_2 - \omega_2\varphi_1) \equiv F(\theta)$. Thus, the stationary solution can be written as

$$p^0 = \frac{\upsilon(\xi)\, F(\theta)}{4\pi^2} \tag{15}$$

where $F(\theta)$ is yet to be determined. In the work of Arnold *et al.*,[3] both p^0 and L^0 are functions of the same independent variables. However, in the present case L^0 is only a function of $(\varphi_1, \varphi_2, \xi)$, whereas p^0 depends, in general, on φ_1, φ_2, θ, and ξ. Thus, one cannot completely determine p^0 by solving the $O(1)$ equation. In order to calculate p^0 completely, it is necessary to consider the ε-order Poisson equation along with its adjoint problem, i.e.,

$$L^{0*}p^1 = -L^{1*}p^0 \qquad \text{and} \qquad L^0 u = 0 \tag{16}$$

The solution for p^0 in Eq. (12b) requires that $L^{1*}p^0$ satisfy the solvability condition

$$\langle L^{1*}p^0, u\rangle = 0 \qquad \forall u \in \ker(L^0) \tag{17}$$

Due to the assumption on G and the periodic boundary conditions, $\ker(L^0) = \{C(\theta)\colon C \text{ is an arbitrary function}\}$ and the solvability condition reduces to

$$\int_0^{\pi/2} C(\theta) \int_M f(\xi)\, v(\xi)\, d\xi \times \left\{ \int_0^{2\pi} \int_0^{2\pi} \left[\frac{\partial}{\partial\theta}(q_2 F) + F \sum_{i=1}^{2} \frac{\partial h_i}{\partial \varphi_i} \right] d\varphi_1\, d\varphi_2 \right\} d\theta = 0 \tag{18}$$

which is automatically satisfied since the expression in braces is identically zero. In addition, for arbitrary $F(\theta)$ the inner product $\langle Q^1, p^0\rangle = 0$. This implies that the leading-order approximation of the Lyapunov exponent is

$$\lambda^{\varepsilon} = \varepsilon^2(\langle Q^2, p^0\rangle + \langle Q^1, p^1\rangle) + O(\varepsilon^3) \tag{19}$$

It is clear that both p^0 and p^1 are required to glean any information on λ^{ε}. The solvability condition for Eq. (16), being identically satisfied, yields no information on $F(\theta)$, thus leaving p^0 undetermined. This further implies that a solution of Eq. (16) for p^1 exists for arbitrary $F(\theta)$. However, it will be shown that the solvability condition for Eq. (11c) will provide $F(\theta)$ yielding a unique p^0 and, in turn, p^1. In order to solve for p^1, Eq. (16) may be written as

$$\begin{aligned}\left(G^* - \sum_{i=1}^{2} \omega_i \frac{\partial}{\partial \varphi_i}\right) p^1 &= \frac{f(\xi)\, v(\xi)}{4\pi^2} [-F_1(\theta) \sin 2\varphi_1 + F_2(\theta) \sin 2\varphi_2 \\ &\quad + F^+(\theta) \sin \varphi^+ - F^-(\theta) \sin \varphi^-] \\ &= \frac{f(\xi)\, v(\xi)}{4\pi^2} R(\varphi_1, \varphi_2, \theta)\end{aligned} \tag{20}$$

where

$$F_1(\theta) = \left[\frac{1}{4} \sin 2\theta \frac{dF}{d\theta} + \left(1 + \frac{1}{2} \cos 2\theta\right) F\right] p_{11}$$

$$F_2(\theta) = \left[\frac{1}{4} \sin 2\theta \frac{dF}{d\theta} - \left(1 - \frac{1}{2} \cos 2\theta\right) F\right] p_{22}$$

$$\begin{aligned}F^{\delta}(\theta) = \frac{1}{2} \Big\{ &(p_{21} \cos^2\theta - \delta p_{12} \sin^2\theta) \frac{dF}{d\theta} \\ &- [(\sin 2\theta + \cot\theta)\, p_{21} + \delta(\sin 2\theta + \tan\theta)\, p_{12}] F \Big\}\end{aligned}$$

$\varphi^{\delta}=\varphi_1+\delta\varphi_2$ and $\delta=\pm 1$. Various methods are available for solving the above equation. However, in order to make use of the transient density $g(x, t; \eta, 0)$, which is a solution of

$$\frac{\partial g}{\partial t}=G^* g, \qquad g(\xi, 0; \eta, 0)=\delta(\xi-\eta) \tag{21}$$

the approach of Arnold *et al.*[3] will be adopted. The solution p^1 may be considered as the limiting steady-state solution of the following transient problem with zero initial condition:

$$\left(\frac{\partial}{\partial t}+\sum_{i=1}^{2}\omega_i\frac{\partial}{\partial\varphi_i}-G^*\right)p_t^1=\frac{f(\xi)\,v(\xi)}{4\pi^2}R(\varphi_1, \varphi_2, \theta) \tag{22}$$

Making use of the transformation

$$\tau=\frac{1}{2}\left[t+\frac{1}{2}\left(\frac{\varphi_1}{\omega_1}+\frac{\varphi_2}{\omega_2}\right)\right]$$

$$s=\frac{1}{2}\left[t-\frac{1}{2}\left(\frac{\varphi_1}{\omega_1}+\frac{\varphi_2}{\omega_2}\right)\right]$$

$$\gamma=\omega_1\varphi_2-\omega_2\varphi_1$$

in Eq. (22) yields

$$\left(\frac{\partial}{\partial\tau}-G^*\right)p_\tau^1=\frac{f(\xi)\,v(\xi)}{4\pi^2}R(\varphi_1(\tau, s, \gamma), \varphi_2(\tau, s, \gamma), \theta)$$

whose solution can be written as

$$p_\tau^1(s, \gamma, \theta, \xi)=\frac{1}{4\pi^2}\int_0^\tau\left[\int_M f(\eta)\,v(\eta)\,g(\xi, T; \eta, 0)\,d\eta\right]$$
$$\times R(\varphi_1(\tau-T, s, \gamma), \varphi_2(\tau-T, s, \gamma), \theta)\,dT$$

The final form of p^1 can be written reverting to the original coordinates $\{t, \varphi_1, \varphi_2\}$ and taking the limit as $t\to\infty$:

$$p^1(\varphi_1, \varphi_2, \theta, \xi)=\frac{1}{4\pi^2}\int_0^\infty H(\varphi_1, \varphi_2, \theta, T)\,K(\xi, T)\,dT \tag{23}$$

where

$$H(\varphi_1, \varphi_2, \theta, \tau)=-F_1(\theta)\sin(2\omega_1\tau-2\varphi_1)+F_2(\theta)\sin(2\omega_2\tau-2\varphi_2)$$
$$+F^+(\theta)\sin(\Omega^+\tau-\varphi^+)-F^-(\theta)\sin(\Omega^-\tau-\varphi^-)$$

$$K(\xi, \tau)=\int_M f(\eta)\,v(\eta)\,g(\xi, \tau; \eta, 0)\,d\eta$$

and $\Omega^\delta = \omega_1 + \delta\omega_2$. Substituting p^0 and p^1 in the expression for the maximal Lyapunov exponent yields

$$\lambda^\varepsilon = -\int_0^{\pi/2} (\eta_1 \cos^2\theta + \eta_2 \sin^2\theta)\, F(\theta)\, d\theta$$
$$+ \frac{1}{4\pi^2} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2\pi} \left[\int_0^\infty \int_M f(\xi)\, K(\xi, \tau)\, d\xi\, H(\varphi_1, \varphi_2, \theta, \tau)\, d\tau \right]$$
$$\times q_1(\varphi_1, \varphi_2, \theta)\, d\varphi_1\, d\varphi_2\, d\theta \tag{24}$$

Making use of the fact that the correlation of $f(\xi)$ is given by

$$R(\tau) = \int_M f(\xi)\, K(\xi, \tau)\, d\xi = \int_M \int_M f(\xi)\, f(\eta)\, \upsilon(\eta)\, g(\xi, \tau, \eta, 0)\, d\xi\, d\eta$$

and the cosine spectrum is defined as

$$S(\omega) = 2 \int_0^\infty R(\tau) \cos \omega\tau\, d\tau$$

the Lyapunov exponent becomes

$$\lambda^\varepsilon = \int_0^{\pi/2} \left[\lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta + \Psi^2(\theta) + \frac{1}{8} p_{21} p_{12} \sum_\delta \delta S(\Omega^\delta) \right] F(\theta)\, d\theta \tag{25}$$

where

$$\Psi^2(\theta) = \frac{1}{32} \left[\sum_{i=1}^{2} p_{ii}^2 S(2\omega_i) \right] \sin^2 2\theta + \frac{1}{32} \sum_\delta \left[(p_{21} + \delta p_{12})^2 \cos^2 2\theta \right.$$
$$\left. + (p_{21} - \delta p_{12})^2 + 2(p_{21}^2 - p_{12}^2) \cos 2\theta \right] S(\Omega^\delta)$$

$$\lambda_i = -\eta_i + \frac{1}{8} p_{ii} S(2\omega_i)$$

One may easily verify that $\psi^2(\theta) \geqslant 0$. It now remains to determine $F(\theta)$. To this end, the solvability condition of Eq. (11c) is needed. Inserting Eqs. (15) and (23) into (11c) yields

$$\left(G^* - \sum_{i=1}^{2} \omega_i \frac{\partial}{\partial \varphi_i} \right) p^2 = \chi_0(\varphi_1, \varphi_2, \theta, \xi) + \chi_1(\varphi_1, \varphi_2, \theta, \xi) \tag{26}$$

where

$$\chi_0(\varphi_1, \varphi_2, \theta, \xi) = \frac{\upsilon(\xi)}{4\pi^2}\left[F(\theta)\left(\frac{\partial \tilde{q}_2}{\partial \theta} + \sum_{i=1}^{2} \frac{\partial \tilde{h}_i}{\partial \varphi_i}\right) + \tilde{q}_2 \frac{dF}{d\theta}\right]$$

$$\chi_1(\varphi_1, \varphi_2, \theta, \xi) = \frac{f(\xi)}{4\pi^2}\int_0^\infty \left[H(\varphi_1, \varphi_2, \theta, T)\left(\frac{\partial q_2}{\partial \theta} + \sum_{i=1}^{2} \frac{\partial h_i}{\partial \varphi_i}\right)\right.$$
$$\left. + q_2 \frac{\partial H}{\partial \theta} + \sum_{i=1}^{2} h_i \frac{\partial H}{\partial \varphi_i}\right] K(\xi, T)\, dT$$

The solvability condition reduces to

$$\int_0^{\pi/2} C(\theta) \int_M \int_0^{2\pi} \int_0^{2\pi} [\chi_0(\varphi_1, \varphi_2, \theta, \xi)$$
$$+ \chi_1(\varphi_1, \varphi_2, \theta, \xi)]\, d\varphi_1\, d\varphi_2\, d\xi\, d\theta = 0 \tag{27}$$

For this condition to hold for arbitrary $C(\theta)$ one must have

$$\hat{F}_0(\theta) + \hat{F}_1(\theta) = 0 \tag{28}$$

where

$$\hat{F}_0(\theta) = \int_M \int_0^{2\pi} \int_0^{2\pi} \chi_0(\varphi_1, \varphi_2, \theta, \xi)\, d\varphi_1\, d\varphi_2\, d\xi$$

and

$$\hat{F}_1(\theta) = \int_M \int_0^{2\pi} \int_0^{2\pi} \chi_1(\varphi_1, \varphi_2, \theta, \xi)\, d\varphi_1\, d\varphi_2\, d\xi$$

The first integral, $\hat{F}_0$, reduces to

$$\hat{F}_0(\theta) = \frac{1}{2}(\eta_1 - \eta_2)\frac{\partial}{\partial \theta}[F(\theta) \sin 2\theta]$$

and the second integral reduces to

$$\hat{F}_1(\theta) = -\frac{1}{16}\sum_{i=1}^{2} p_{ii} \frac{\partial}{\partial \theta}[F_i(\theta) \sin 2\theta]\, S(2\omega_i)$$
$$-\frac{1}{8}\sum_{\delta} p_{ii} \frac{\partial}{\partial \theta}[F^\delta(\theta)(p_{21} \cos^2 \theta - \delta p_{12} \sin^2 \theta)]\, S(\Omega^\delta)$$

Combining the above results for $\hat{F}_0$ and $\hat{F}_1$ in Eq. (28) yields a second-order ordinary differential equation for $F(\theta)$,

$$-\frac{d}{d\theta}[\Phi(\theta)\, F(\theta)] + \frac{1}{2}\frac{d^2}{d\theta^2}[\Psi^2(\theta)\, F(\theta)] = 0 \tag{29}$$

where

$$\Phi(\theta) = -\frac{1}{2}\left[(\lambda_1 - \lambda_2) + \frac{1}{16}(p_{21}^2 - p_{12}^2)\sum_{\delta} S(\Omega^\delta)\right]\sin 2\theta$$
$$+\frac{1}{64}\left[\sum_{i=1}^{2} p_{ij}^2 S(2\omega_i) - \sum_{\delta}(p_{21} + \delta p_{12})^2 S(\Omega^\delta)\right]\sin 4\theta$$
$$+\frac{1}{8}\left\{\sum_{\delta} S(\Omega^\delta)\right\}\frac{p_{21}^2\cos^4\theta - p_{12}^2\sin^4\theta}{\sin 2\theta}$$

In order to complete the analysis, the rotation number for each degree of freedom is calculated. By similar arguments used in deriving Eq. (12b), asymptotic expansion of the rotation numbers can be constructed by replacing Q^ε by H_i^ε, which yields

$$\alpha_i^\varepsilon = \omega_i + \varepsilon\langle H_i^1, p^0\rangle + \varepsilon^2[\langle H_i^2, p^0\rangle + \langle H_i^1, p^1\rangle] + O(\varepsilon^2)$$

Substituting for p^0 and p^1 in the above equations and making use of the fact that $f(\xi)$ has zero mean yields

$$\alpha_i^\varepsilon = \omega_i + \frac{\varepsilon^2}{4\pi^2}\int_0^{\pi/2}\int_0^{2\pi}\int_0^{2\pi}\left[\int_0^\infty \int_M f(\xi)\, K(\xi, \tau)\, d\xi\, H(\varphi_1, \varphi_2, \theta, \tau)\, d\tau\right]$$
$$\times h_i(\varphi_1, \varphi_2, \theta)\, d\varphi_1\, d\varphi_2\, d\theta$$

As in the calculation of Lyapunov exponents, making use of the definitions of correlation and the sine spectrum of $f(\xi)$ results in

$$\alpha_i = \omega_i - \int_0^{\pi/2}\frac{\varepsilon^2}{8}\{p_{ii}^2\Gamma(2\omega_i) + p_{12}p_{21}[\Gamma(\Omega^+) + (-1)^i\Gamma(\Omega^-)]\}$$
$$\times F(\theta)\, d\theta - \varepsilon^2\hat{\alpha}_i \tag{30}$$

where

$$\hat{\alpha}_1 = p_{12}^2 \lim_{\theta\to\pi/2}[\sec\theta\, F(\theta)]$$

$$\hat{\alpha}_2 = p_{21}^2 \lim_{\theta\to 0}[\csc\theta\, F(\theta)]$$

$$\Gamma(\omega) = \int_0^\infty R(\tau)\sin\omega\tau\, d\tau$$

In summary, the above analysis indicates that in order to determine the first nonvanishing Lyapunov exponent, which for this case is of the order ε^2, it is essential that one determines both p^0 and p^1. However, the explicit form of both p^0 and p^1 in turn depends on an arbitrary function $F(\theta)$ which is normalized to one. The equation governing $F(\theta)$ is determined from the solvability condition for p^2, and once Eq. (29) is solved for $F(\theta)$ with the appropriate boundary conditions, the maximal Lyapunov exponent and the rotation numbers can be calculated from Eqs. (25) and (30), respectively.

4. EVALUATION OF SOLUTIONS

The procedure presented so far has not restricted p^0 to be a smooth function of θ. In general, it is possible to have singularities in θ; thus, some justification is needed in order to ensure that p^0 is a bounded positive density. It is clear from the form of the diffusion term that there may be singularities in the open interval $(0, \pi/2)$, i.e., when $\sum_i p_{ii} S(2\omega_i) = 0$, singularities exist for $p_{12} p_{21} > 0$ if $S(\Omega^-) = 0$ and $\cos 2\theta = (p_{12} - p_{21})/(p_{12} + p_{21})$, and for $(p_{12} p_{21}) < 0$ if $S(\Omega^+) = 0$ and $\cos 2\theta = (p_{12} + p_{21})/(p_{12} - p_{21})$. Only the nonsingular cases will be considered. The drift and diffusion terms of Eq. (29) are conveniently written in the form

$$\begin{aligned} \Psi^2(\theta) &= A\cos^2 2\theta + B\cos 2\theta + C \\ \Phi(\theta) &= -\tfrac{1}{2}(\lambda_1 - \lambda_2)\sin 2\theta + \Psi^2(\theta)\cot 2\theta \end{aligned} \tag{31}$$

where

$$A = \frac{1}{32}\left[(p_{21} + p_{12})^2 S(\Omega^+) + (p_{21} - p_{12})^2 S(\Omega^-) - \sum_{i=1}^{2} p_{ii}^2 S(2\omega_i)\right]$$

$$B = \frac{1}{16}(p_{21} + p_{12})(p_{21} - p_{12})[S(\Omega^+) + S(\Omega^-)]$$

$$C = \frac{1}{32}\left[(p_{21} - p_{12})^2 S(\Omega^-) + (p_{21} + p_{12})^2 S(\Omega^+) + \sum_{i=1}^{2} p_{ii}^2 S(2\omega_i)\right]$$

It is clear from the transformation [Eq. (5)] that $\theta = 0$ implies $r_1 = 0$, and $\theta = \pi/2$ implies $r_2 = 0$. It is clear physically that unless the coupling coefficients p_{21} and p_{12} are both zero, it is not possible to have a solution with either r_1 or r_2 identically zero. Thus $F(\theta)$ will not represent a point mass at $\theta = 0$ or $\theta = \pi/2$. Moreover, this assertion can also be justified by applying the Feller classification based on scale and speed measures for the $F(\theta)$ process. It can easily be shown that $(\varphi_i, \theta = 0)$ and $(\varphi_i, \theta = \pi/2)$ are entrance boundaries and the stationary density function is therefore given by

$$F(\theta) = C\frac{\sin 2\theta}{\psi^2(\theta)} \exp\left\{-(\lambda_1 - \lambda_2)\left[\beta(\theta) - \frac{1}{2}\beta\left(\frac{\pi}{2}\right)\right]\right\} \tag{32}$$

where

$$C=\frac{1}{2}(\lambda_1-\lambda_2)\operatorname{csch}\left[\frac{1}{2}(\lambda_1-\lambda_2)\,\beta\left(\frac{\pi}{2}\right)\right], \qquad \beta(\theta)=\int_0^\theta \frac{\sin 2\eta}{\psi^2(\eta)}\,d\eta$$

Substituting expression (32) for $F(\theta)$ into Eq. (25) leads to the following expression for the maximal Lyapunov exponent:

$$\lambda^\varepsilon=\frac{\varepsilon^2}{2}\left\{\lambda_1+\lambda_2+(\lambda_1-\lambda_2)\coth\left[\frac{1}{2}(\lambda_1-\lambda_2)\,\beta\left(\frac{\pi}{2}\right)\right]\right.$$
$$\left.+\frac{1}{4}p_{21}p_{12}[S(\Omega^+)-S(\Omega^-)]\right\} \tag{33}$$

Similarly, substitution of $F(\theta)$ into the expression for the rotation number leads to

$$\alpha_i=\omega_i-\frac{\varepsilon^2}{8}\left\{p_{ii}^2\Gamma(2\omega_i)+p_{12}p_{21}[\Gamma(\Omega^+)+(-1)^i\,\Gamma(\Omega^-)]\right.$$
$$+4(\lambda_1-\lambda_2)\operatorname{csch}\left[\frac{1}{2}(\lambda_1-\lambda_2)\,\beta\left(\frac{\pi}{2}\right)\right]\exp\left[\frac{(-1)^i}{2}(\lambda_1-\lambda_2)\,\beta\left(\frac{\pi}{2}\right)\right]$$
$$\left.\times\frac{\Gamma(\Omega^+)-(-1)^i\,\Gamma(\Omega^-)}{S(\Omega^+)+S(\Omega^+)}\right\}$$

Defining a and b in terms of cosine spectra at $2\omega_i$ and $\omega_1\pm\omega_2$ and stochastic coefficients as

$$a=\tfrac{1}{16}\sum_{i=1}^{2}p_{ii}^2S(2\omega_i)-\tfrac{1}{4}p_{21}p_{12}S(\Omega^+)$$
$$b=\tfrac{1}{16}\sum_{i=1}^{2}p_{ii}^2S(2\omega_i)+\tfrac{1}{4}p_{21}p_{12}S(\Omega^-)$$

we find the expression for $\beta(\pi/2)$:

$$\beta\left(\frac{\pi}{2}\right)=\begin{cases}\dfrac{1}{(ab)^{1/2}}\ln\left|\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right|, & ab>0\\[2ex] \dfrac{2}{a+b}, & ab=0\\[2ex] \dfrac{1}{(-ab)^{1/2}}\tan^{-1}\left(\dfrac{2(-ab)^{1/2}}{a+b}\right), & ab<0\end{cases}$$

When the coupling terms are identically zero (i.e., $p_{21} \to 0$ and $p_{12} \to 0$), it is evident that $ab > 0$ and $\beta(\pi/2) \to \infty$. Thus, Eq. (33) yields $\lambda = \lambda_1$ if $\lambda_1 > \lambda_2$ and $\lambda = \lambda_2$ if $\lambda_2 > \lambda_1$. Since λ_1 and λ_2 are the top Lyapunov exponents respectively for the first and second decoupled oscillators, the above result confirms the fact that the largest Lyapunov exponent of the system is given by Eq. (32).

5. CONCLUSION

In this paper, a perturbative method introduced by Arnold *et al.*[3] is extended to calculate the top Lyapunov exponent and rotation numbers of a coupled, two-degree-of-freedom system parametrically excited by real noise. The asymptotic analysis presented in this paper leads to a sequence of linear Poisson equations to be solved at each order in ε. To determine the solution completely at any particular order requires that a solvability condition be met at the subsequent order in ε. At the leading order, this solvability condition leads to the integration of the right-hand side of the first order of the Poisson equation with respect to φ_1, φ_2, and ξ. Thus, it is natural that the equation for diffusion in θ is similar to that obtained using stochastic averaging. It is also clear from this analysis that stochastic averaging yields the first term in the asymptotic expansion of the Lyapunov exponent. Furthermore, this method can easily be extended to obtain the second-order approximation of the Lyapunov exponent.

The maximal Lyapunov exponent yields the stability boundary in the system parameter space and hence yields the bifurcation points in this space. Even though a stochastic bifurcation occurs when λ^ε passes through zero, the method presented in this paper does not indicate how many exponents pass through zero simultaneously. In addition, the rotation number for each degree of freedom is obtained by examining the winding rate of $\tan^{-1}(x_{2i}/x_{2i-1})$, as opposed to the case of white noise excitation, in which the ε^2-order rotation number is identically zero, the ε^2-order correction in the real noise case is given in terms of the sine spectra at $2\omega_i$ and $\omega_1 \pm \omega_2$.

APPENDIX

It will be shown that if f and g are two C^1 functions which satisfy

$$f(x + m\omega_1 + n\omega_2)\, g(y + m\omega_1 - n\omega_2) = f(x)\, g(y) \qquad \forall x,\, y \in R, \quad m, n \in Z \tag{A1}$$

where ω_1 and ω_2 are noncommensurable, then f and g must be constant. To demonstrate this, Eq. (A1) may be rewritten as

$$\frac{f(x+m\omega_1+n\omega_2)}{f(x)}=\frac{g(y)}{g(y+m\omega_1-n\omega_2)}=\lambda(m,n) \tag{A2}$$

which prompts the definition of the following functions:

$$F(x,z)=\frac{f(x+z)}{f(x)}, \qquad G(y,z)=\frac{g(y)}{g(y+z)} \tag{A3}$$

satisfying $F(x,0)=G(y,0)=1$.

The first step is to show that F and G are functions of z only. To accomplish this, one notes that $\forall z \in R$, there exists a sequence of integers $\{n_i, m_i\}_{i=1}^{\infty}$ for which the following limit holds:

$$\lim_{i\to\infty} (m_i\omega_1+n_i\omega_2)=z$$

This follows from the noncommensurability of ω_1 and ω_2. For arbitrary $x_1, x_2 \in R$, one now obtains

$$\begin{aligned}
&F(x_1,z)-F(x_2,z)\\
&\quad=F(x_1, \lim_{i\to\infty}(m_i\omega_1+n_i\omega_2))-F(x_2, \lim_{i\to\infty}(m_i\omega_1+n_i\omega_2))\\
&\quad=\lim_{i\to\infty} F(x_1, m_i\omega_1+n_i\omega_2)-F(x_2, m_i\omega_1+n_i\omega_2)\\
&\quad=\lim_{i\to\infty}\left(\frac{f(x_1+m_i\omega_1+n_i\omega_2)}{f(x_1)}-\frac{f(x_2+m_i\omega_1+n_i\omega_2)}{f(x_2)}\right)\\
&\quad=\lim_{i\to\infty}(\lambda(m_i,n_i)-\lambda(m_i,n_i))=0
\end{aligned}$$

Hence, one may write $F(x,z)\equiv\hat{F}(z)$, with a similar argument for G leading to $G(y,z)\equiv\hat{G}(z)$.

The second step is to show that $\hat{F}$ and $\hat{G}$ are constant. To this end, inserting these results into (A3) leads to

$$f(x+z)=f(x)\,\hat{F}(z), \qquad g(y+z)=\frac{g(y)}{\hat{G}(z)} \tag{A4}$$

with $\hat{F}(0)=\hat{G}(0)=1$. Comparing the x-derivative and z-derivative of the first of Eqs. (A4) leads to

$$f'(x)\,\hat{F}(z)=f(x)\,\hat{F}'(z)$$

from which it follows that

$$\frac{f'(x)}{f(x)} = \frac{\hat{F}'(z)}{\hat{F}(z)} = \mu_f \quad \text{(const)}$$

which in turn leads to

$$f(x) = C_F e^{\mu_F x}, \qquad \hat{F}(z) = e^{\mu_F z}$$

where C_F is a constant. Similarly,

$$g(y) = C_G e^{\mu_G y}, \qquad \hat{G}(z) = e^{\mu_G z}$$

Using the expressions for $\hat{F}$ and $\hat{G}$ in Eqs. (A2) and (A3) yields

$$e^{\mu_F(m\omega_1 + n\omega_2) - \mu_G(m\omega_1 - n\omega_1)} = 1$$

For this to hold for all $m, n \in Z$, it follows that $\mu_F = \mu_G = 0$ and hence $f(x) = C_F$, $g(y) = C_G$.

ACKNOWLEDGMENTS

The authors would like to acknowledge Prof. Wolfgang H. Kliemann of Iowa State University and Prof. Mark Pinsky of Northwestern University for various helpful discussions. N. S. N. would like to acknowledge the support of AFOSR through grant 93-0063 and the National Science Foundation through grant MSS-90-57437 PYI.

REFERENCES

1. S. T. Ariaratnam and W. C. Xie, *J. Appl. Mech.* **59**(3):664 (1992).
2. L. Arnold and V. Wihstutz, *Lyapunov Exponents* (Lecture Notes in Mathematics, No. 1186; Springer-Verlag, Berlin, 1986).
3. L. Arnold, G. Papanicolaou, and V. Wihstutz, *SIAM J. Appl. Math.* **46**(3):427 (1986).
4. R. Z. Khasminskii, *Theory Prob. Appl.* **12**(1):144 (1967).
5. F. Kozin, *J. Math. Phys.* **42**:59 (1963).
6. F. Kozin and S. Prodromou, *SIAM J. Appl. Math.* **21**(3):413 (1971).
7. R. R. Mitchell and F. Kozin, *SIAM J. Appl. Math.* **274**(4):571 (1974).
8. K. Nishioka, *Kodai Math. Seminar* **27**:211 (1976).
9. V. I. Oseledec, *Trans. Moscow Math. Soc.* **19**:197 (1968).
10. E. Pardoux and V. Wihstutz, *SIAM J. Appl. Math.* **48**(2):442 (1987).
11. M. A. Pinsky and V. Wihstutz, *Lect. Appl. Math.* **27**:201 (1991).
12. N. Sri Namachchivaya, *J. Sound Vibration* **15**(1):77 (1991).
13. N. Sri Namachchivaya and S. Talwar, *J. Sound Vibration*, to appear (1993).
14. N. Sri Namachchivaya, H. Van Roessel, and S. Talwar, *J. Appl. Mech.*, submitted (1992).